Fluctuations in Statistical Models with Exact Conservation Laws

Large volume limit is not equal to charge or heat bath

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- Introduction: Pion gas with exact Q conservation
- General asymptotic behaviour
- Simulations: Fluctuations as a function of Volume
- The next step: Microcanonical ensemble

Introduction: Neutral Pion Gas

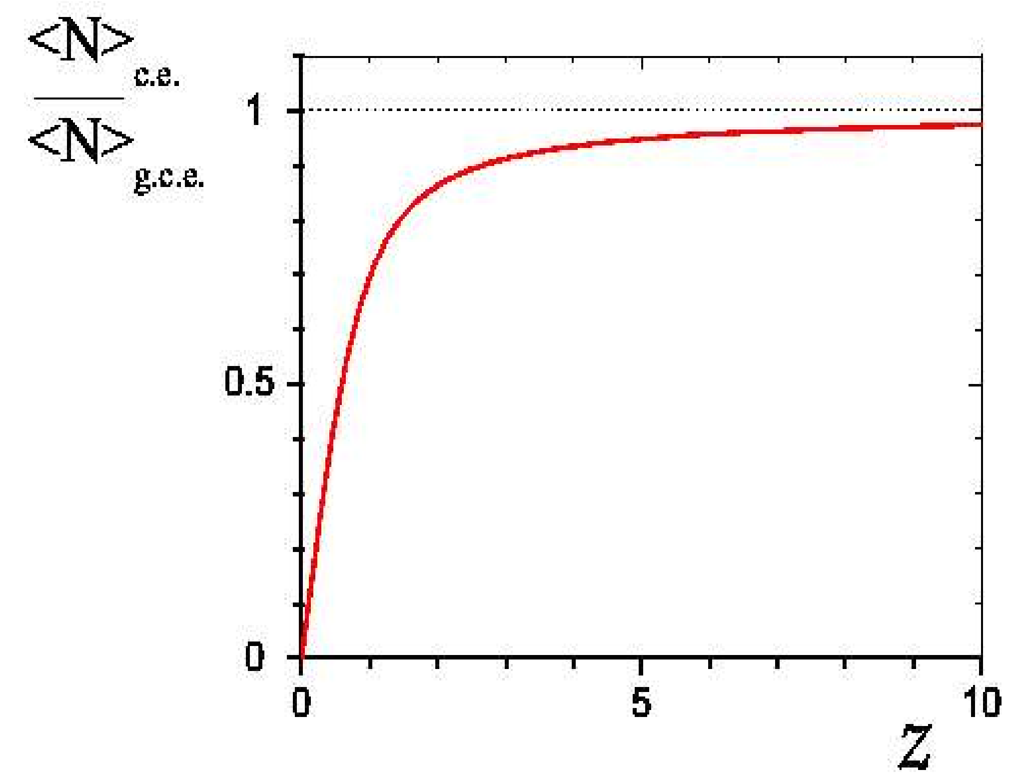
- Mean particle numbers: *Grand Canonical* vs. *Canonical* ensemble

$$\langle N_\pm \rangle_{g.c.e.} = \left(\frac{\partial}{\partial \lambda_\pm} \ln Z_{g.c.e.} \right)_{\lambda_\pm = 1} = z \, .$$

$$\langle N_\pm \rangle_{c.e.} = \left(\frac{\partial}{\partial \lambda_\pm} \ln Z_{c.e.} \right)_{\lambda_\pm = 1} = z \, \frac{I_1(2z)}{I_0(2z)} \, .$$

- Charged particle suppression due to exclusion of total charge fluctuations
- Large volume limit:

$$\frac{I_1(2z)}{I_0(2z)} \rightarrow \lambda$$



Introduction: Neutral Pion Gas

- Useful measure: *The scaled variance*

$$\omega^X \equiv \frac{\langle X^2\rangle - \langle X\rangle^2}{\langle X\rangle}.$$

- For Q=0 pion gas:

$$\omega^{\pm}_{g.c.e.} = \frac{\langle N_\pm^2\rangle_{g.c.e.} - \langle N_\pm\rangle^2_{g.c.e.}}{\langle N_\pm\rangle_{g.c.e.}} = 1\,,$$

$$\omega^{\pm}_{c.e.} = \frac{\langle N_\pm^2\rangle_{c.e.} - \langle N_\pm\rangle^2_{c.e.}}{\langle N_\pm\rangle_{c.e.}} = 1 - z\left[\frac{I_1(2z)}{I_0(2z)} - \frac{I_2(2z)}{I_1(2z)}\right].$$

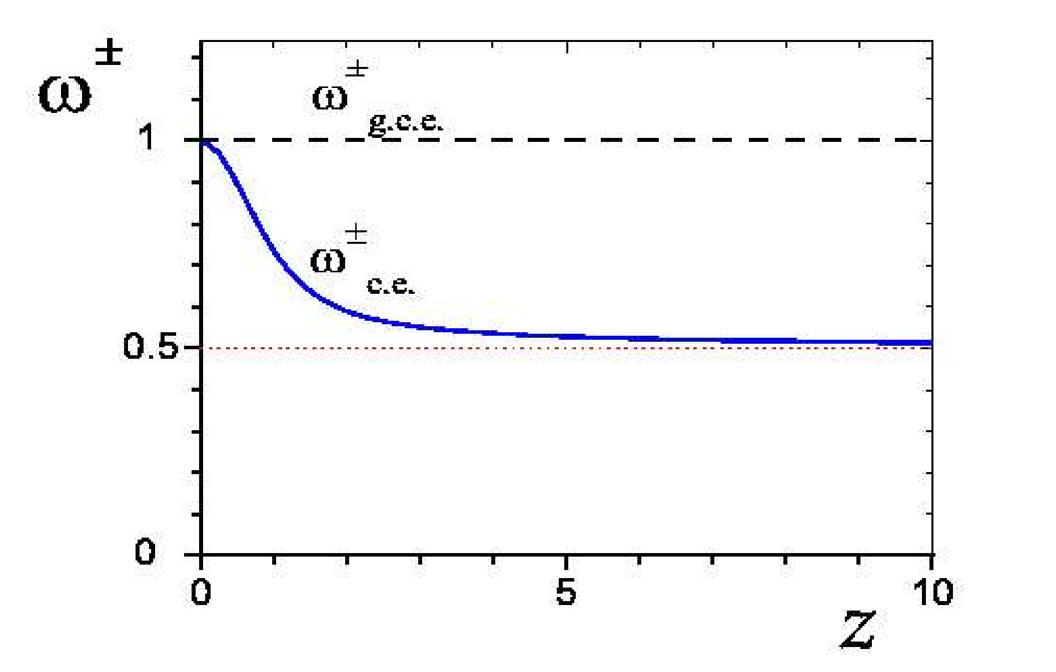
- Limiting cases:

$V \to 0$:

$$\omega^{\pm}_{c.e} \cong 1 - \frac{z^2}{2} \cong 1 = \omega^{\pm}_{g.c.e}\,.$$

$V \to \infty$:

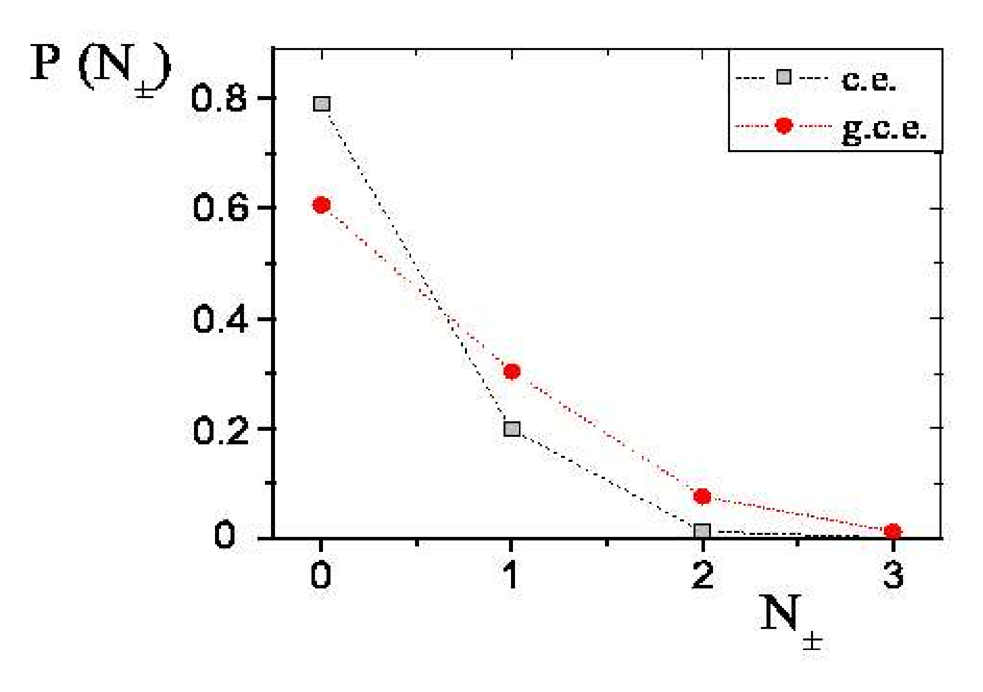
$$\omega^{\pm}_{c.e.} \cong \frac{1}{2} + \frac{1}{8z} \cong \frac{1}{2} = \frac{1}{2}\,\omega^{\pm}_{g.c.e.}\,.$$



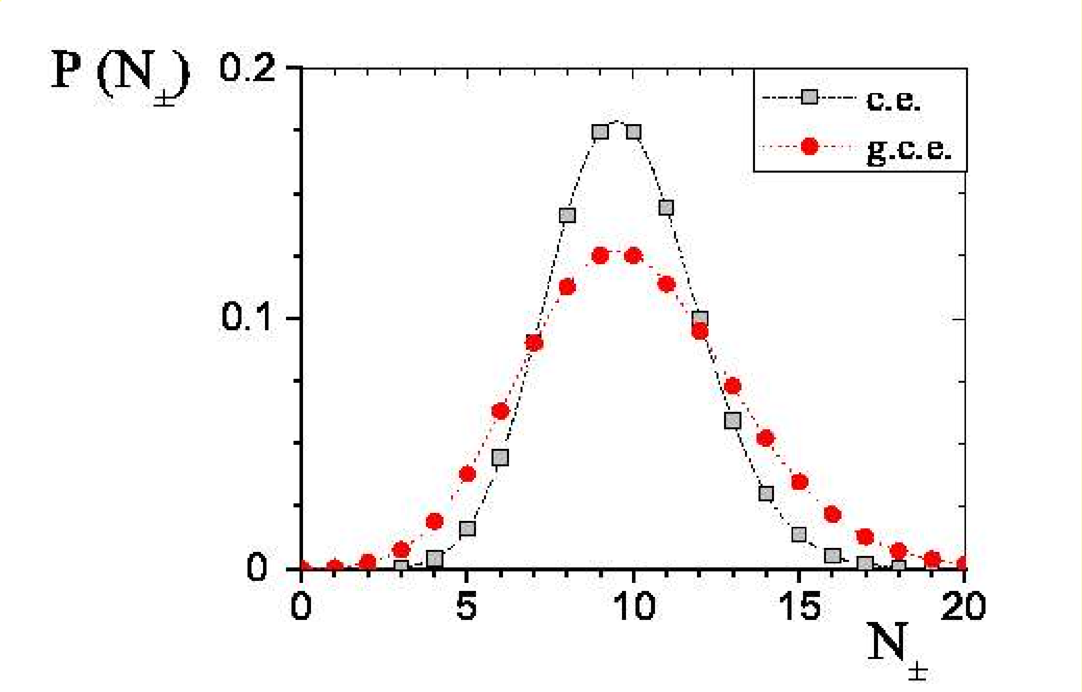
Introduction: Neutral Pion Gas

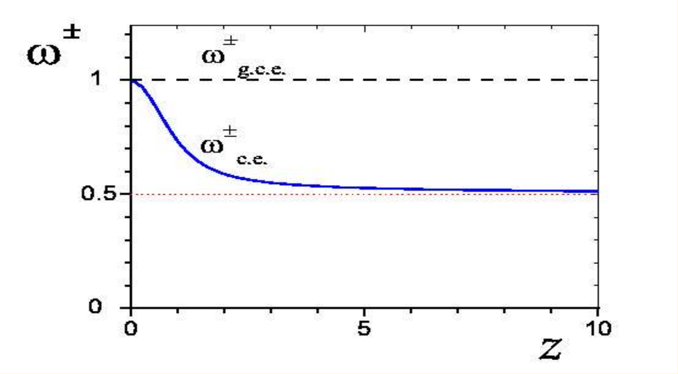
- Multiplicity distributions for negatives (or positives):

$Z = 0.5$



$Z = 10$

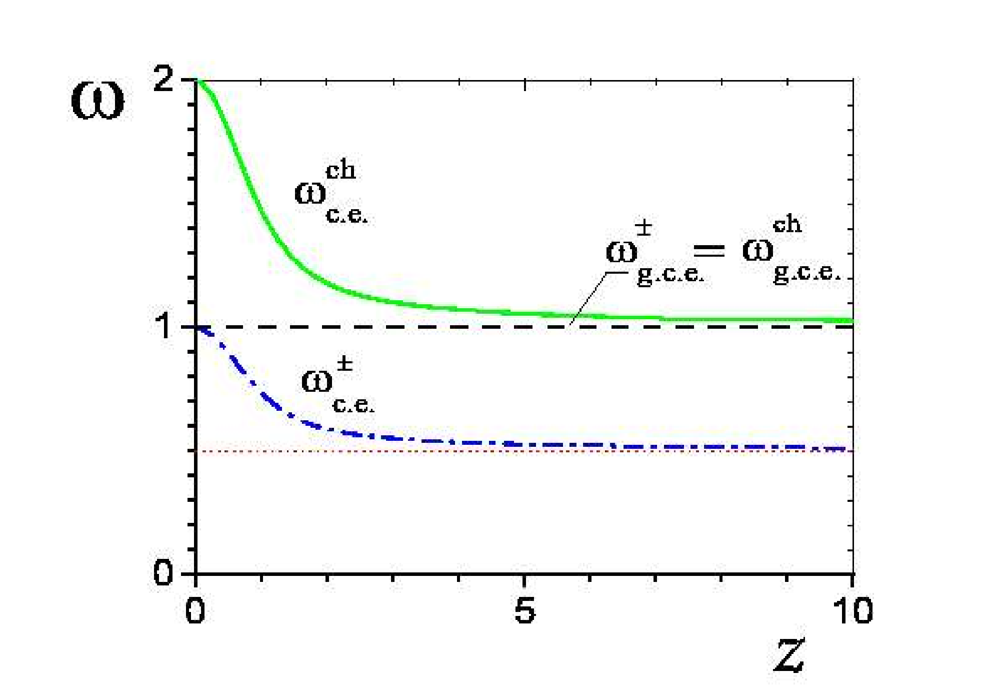


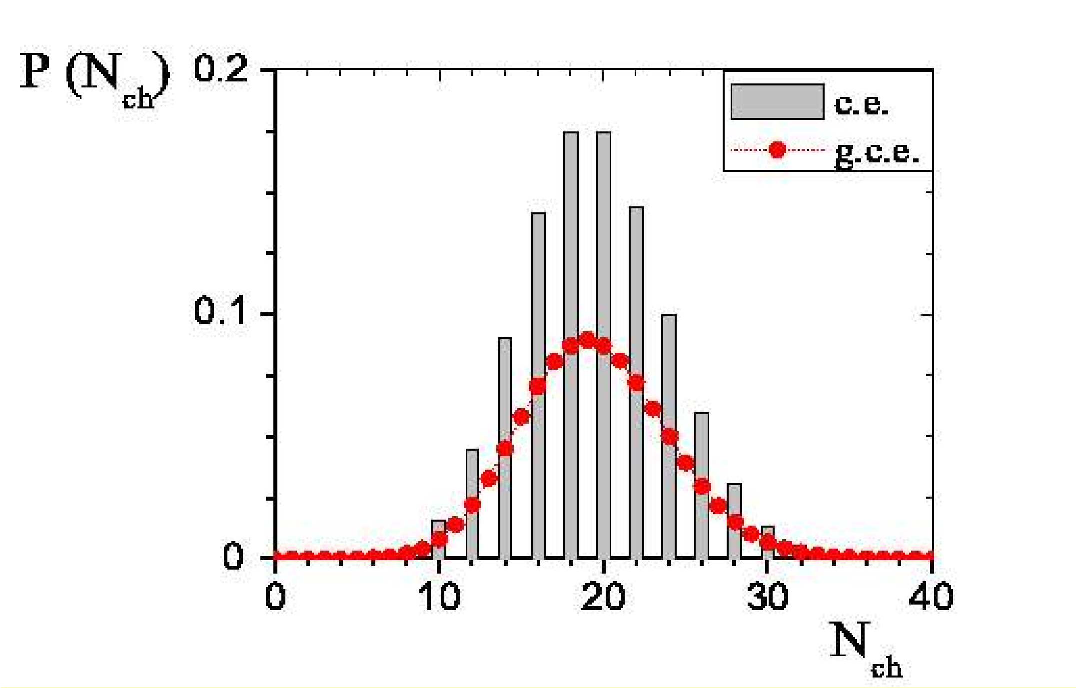


Introduction: Neutral Pion Gas

- Multiplicity distributions for total charged pions:

$Z = 10$





- Note normalization: *Only even number of charged pions allowed*

General asymptotics

- All hadrons up to mass ~1.8 GeV taken into account
- All relevant charges (B,S,Q) conserved exactly
- No analytical solution for all V
- Find large volume limit for partition function:

$$Z_{\vec{Q}}(T,V) = \frac{1}{(2\pi i)^3} \oint dx_B dx_S dx_Q \, x_B^{-B-1} x_S^{-S-1} x_Q^{-Q-1} \exp\left\{ \sum_j z_j^1 x_B^{B_j} x_S^{S_j} x_Q^{Q_j} \right\}.$$

- Saddle point at $\boldsymbol{x}_0 = (\lambda_B, \lambda_S, \lambda_Q)$ Neutral system: $\boldsymbol{x}_0 = (1,1,1)$
- Solution up to order V^{-1}:

$$\begin{aligned} B_{\vec{Q}} \equiv Z_{\vec{Q}}/\alpha = 1 - \frac{1}{2} \Bigg[& (B+1)(B+2) \left(\sum_{i=1}^{3} \frac{\bar{\Lambda}_B^i \bar{\Lambda}_B^j}{h_i} \right) + (S+1)(S+2) \left(\sum_{i=1}^{3} \frac{\bar{\Lambda}_S^i \bar{\Lambda}_S^j}{h_i} \right) \\ & + (Q+1)(Q+2) \left(\sum_{i=1}^{3} \frac{\bar{\Lambda}_S^i \bar{\Lambda}_S^j}{h_i} \right) + 2(B+1)(S+1) \left(\sum_{i=1}^{3} \frac{\bar{\Lambda}_B^i \bar{\Lambda}_S^j}{h_i} \right) \\ & + 2(B+1)(Q+1) \left(\sum_{i=1}^{3} \frac{\bar{\Lambda}_B^i \bar{\Lambda}_Q^j}{h_i} \right) + 2(S+1)(Q+1) \left(\sum_{i=1}^{3} \frac{\bar{\Lambda}_S^i \bar{\Lambda}_Q^j}{h_i} \right) \Bigg] \end{aligned}$$

$$\begin{aligned} B_{\vec{Q}} = 1 - \frac{1}{2} \big[& (B+1)(B+2) a_{BB} + (S+1)(S+2) a_{SS} + (Q+1)(Q+2) a_{QQ} \\ & + 2(B+1)(S+1) a_{BS} + 2(B+1)(Q+1) a_{BQ} + 2(S+1)(Q+1) a_{SQ} \big] \end{aligned}$$

General asymptotics

- The first and second momenta of multiplicity:

$$\langle N\rangle = \sum_j \langle N_j\rangle = \sum_j z_j^1 \frac{Z_{\vec{Q}-\vec{q}_j}}{Z_{\vec{Q}}}$$
$$\langle N^2\rangle = \sum_j \langle N_j\rangle + \sum_j z_j^1 \sum_i z_i^1 \frac{Z_{\vec{Q}-\vec{q}_j-\vec{q}_i}}{Z_{\vec{Q}}}.$$

- Scaled variance:

$$\omega = 1 + \frac{\sum_i \langle N_i\rangle \sum_j z_j^1 \left(\frac{B_{\vec{Q}-\vec{q}_i-\vec{q}_j}}{B_{\vec{Q}-\vec{q}_i}} - \frac{B_{\vec{Q}-\vec{q}_j}}{B_{\vec{Q}}}\right)}{\sum_i \langle N_i\rangle}.$$

- From asymptotic expansion:

$$\begin{aligned}\frac{B_{\vec{Q}-\vec{q}_i-\vec{q}_j}}{B_{\vec{Q}-\vec{q}_i}} - \frac{B_{\vec{Q}-\vec{q}_j}}{B_{\vec{Q}}} &= -(B_iB_ja_{BB} + S_iS_ja_{SS} + Q_iQ_ja_{QQ}) - (B_iS_j + S_iB_j)a_{BS} \\ &\quad -(B_iQ_j + Q_iB_j)a_{BQ} - (S_iQ_j + Q_iS_j)a_{SQ} + O(V^{-2}) \\ &\equiv A_{ij}(V^{-1}) + O(V^{-2}).\end{aligned}$$

No constant term, no dependence on total charges

→ $$\omega = 1 + \frac{\sum_i \langle N_i\rangle \sum_j z_j^1 \left[A_{ij} + O(V^{-2})\right]}{\sum_i \langle N_i\rangle}.$$ *All the volume dependence eliminated!*

- First result: $\lim_{V\to\infty} \omega = 1.$ For multispecies, multicharge Boltzmann gas

General asymptotics

- Quantum gas:

$$\ln(Z_{GC})_j = \sum_{n_j=1}^{\infty} z_j^{(n_j)} \lambda_j^{n_j},$$

$$z_j^{(n_j)}(T,V) = (\mp 1)^{n_j+1} \frac{g_j V}{2\pi^2} T m_j^2 K_2\left(\frac{n_j m_j}{T}\right)$$

$$\omega = \left[\frac{\sum_i \sum_{n_i} n_i^2 z_i^{(n_i)} \frac{Z_{\vec{Q}-n_i\vec{q}_i}}{Z_{\vec{Q}}}}{\sum_i \langle N_i \rangle}\right] + \frac{\sum_i \sum_{n_i} n_i z_i^{(n_i)} \frac{Z_{\vec{Q}-n_i\vec{q}_i}}{Z_{\vec{Q}}} \sum_j \sum_{n_j} n_j z_j^{(n_j)} \left[n_i n_j A_{ij} + O(V^{-2})\right]}{\sum_i \langle N_i \rangle}$$

$\neq 1\,!$

- [] < 1 for baryons and [] > 1 for mesons, applies for GC too

T [MeV]	120			160			180		
statistics	M-B	Q-S	GC	M-B	Q-S	GC	M-B	Q-S	GC
ω_B	0.500	0.500	1.000	0.500	0.500	1.000	0.500	0.500	1.000
ω_S	0.504	0.506	1.004	0.517	0.520	1.006	0.524	0.528	1.007
ω_-	0.502	0.535	1.066	0.509	0.536	1.055	0.512	0.535	1.045
$\omega_{\mathrm{tot.}}$	1	1.061	1.061	1	1.047	1.047	1	1.038	1.038
ω_{π^-}	0.603	0.642	1.088	0.762	0.824	1.119	0.831	0.911	1.131
ω_p	0.892	0.892	1.000	0.931	0.931	0.999	0.942	0.941	0.998

Simulations

- Importance sampling integration:

$$\langle O \rangle \doteq \frac{\sum_{k=1}^{M} O(\{N_j\}^{(k)}) \frac{\Omega_{\{N_j\}}^{(k)}}{\Pi_{\{N_j\}}^{(k)}}}{\sum_{k=1}^{M} \frac{\Omega_{\{N_j\}}^{(k)}}{\Pi_{\{N_j\}}^{(k)}}}$$

$$\Pi_{\{N_j\}} = \prod_{j=1}^{K} \exp[-\nu_j] \frac{\nu_j^{N_j}}{N_j!}$$

- GC Boltzmann distribution is Poissonian: *Fast and easy to sample, close to the physical weight*

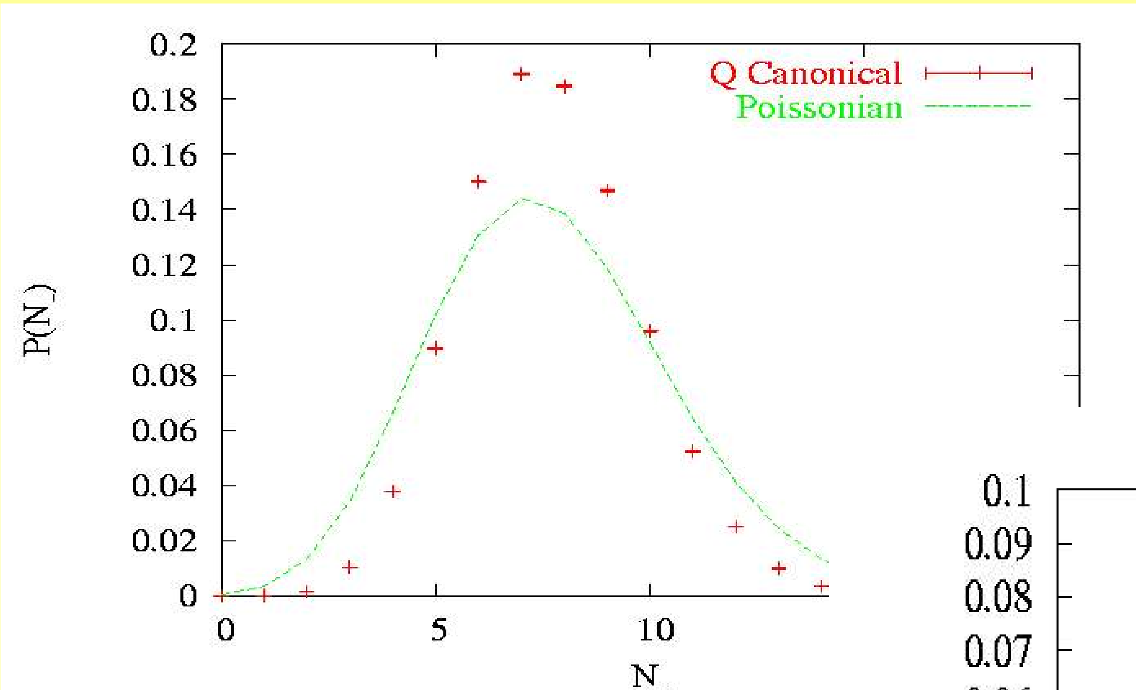
- Physical weight by cluster decomposition:

$$P(\{N_j\}) = \\ = \frac{1}{Z(\mathbf{Q})} \left[\prod_{j=1}^{K} \sum_{\{h_{n_j}\}} \frac{(\mp 1)^{N_j + H_j} z_{j(n_j)}^{H_j}}{\prod_{n_j} n_j^{h_{n_j}} h_{n_j}!} \right] \delta_{\mathbf{Q}, \sum_j N_j \mathbf{q}_j}$$

- Accurate quantum solutions for arbitrary set of system parameters in reasonable time (~ 1 h, desktop machine)

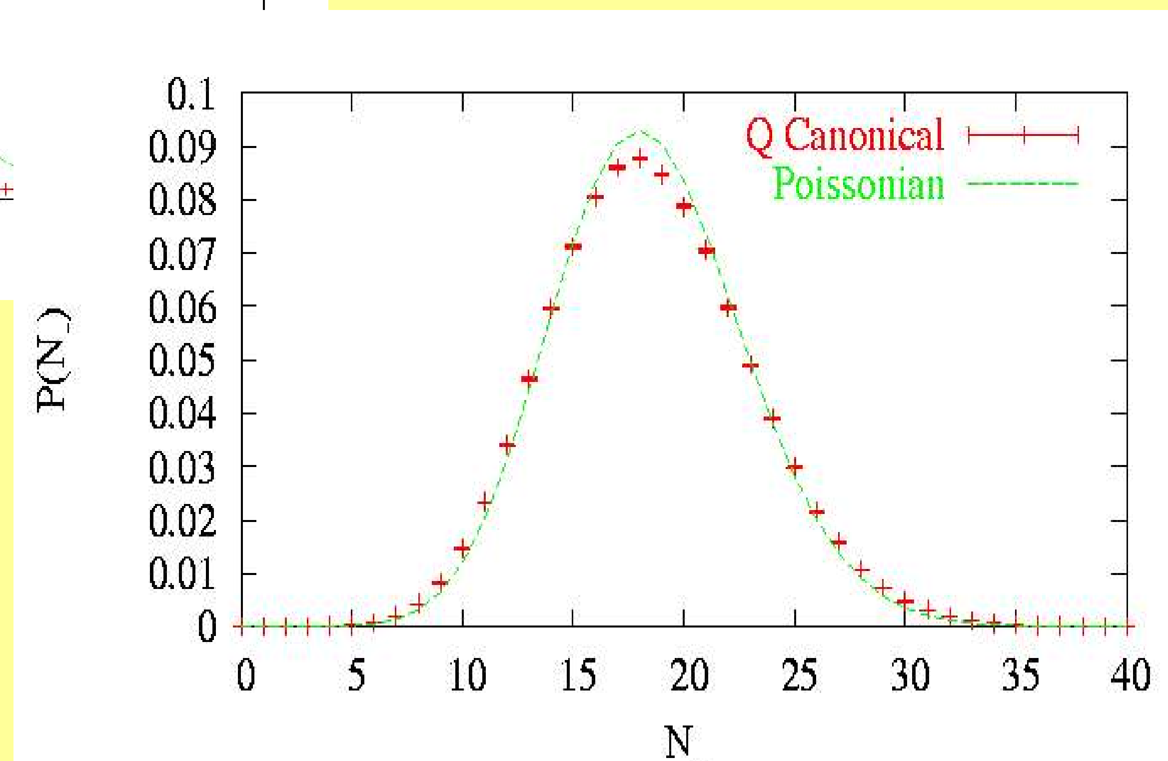
Simulations

- Distribution of negatives in neutral system: T=160 MeV, V=70 fm^3



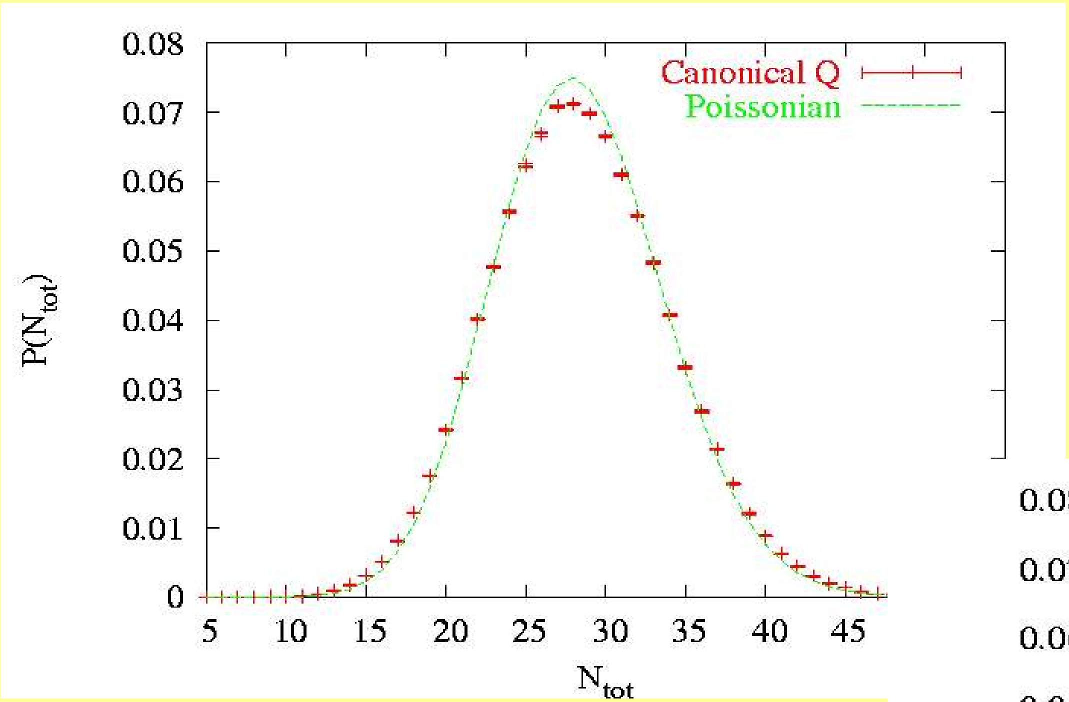
Primaries

Finals after strong decay

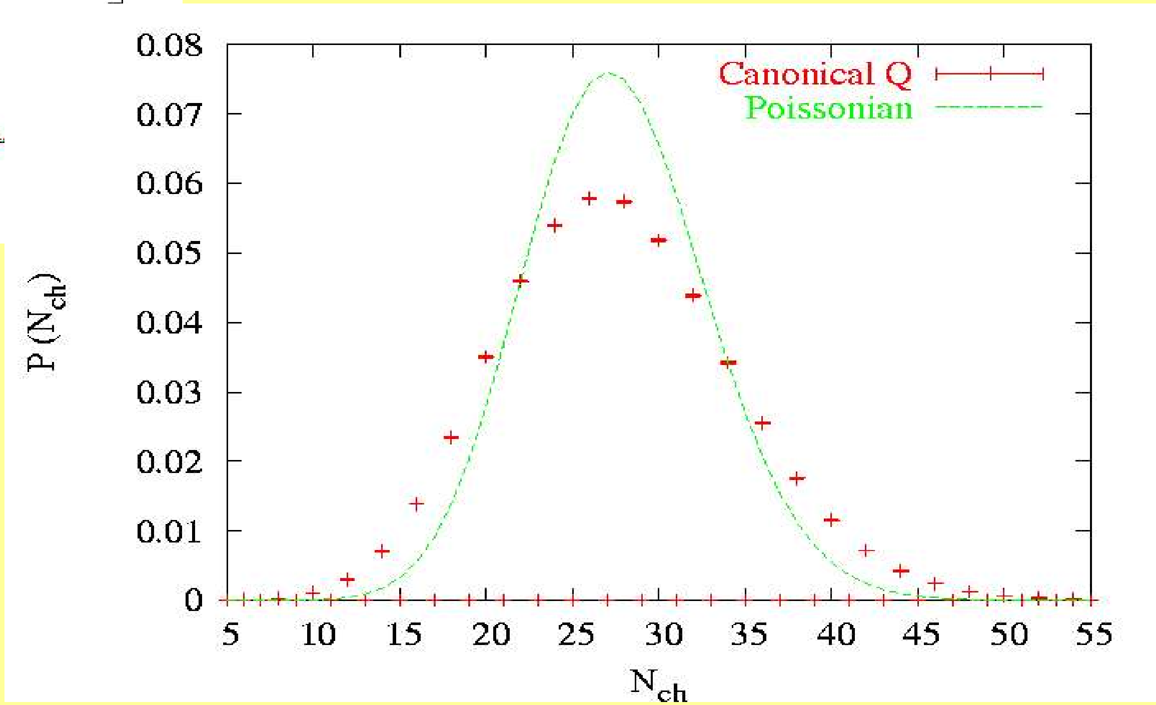


Simulations

- Distributions in neutral system: T=160 MeV, V=70 fm^3



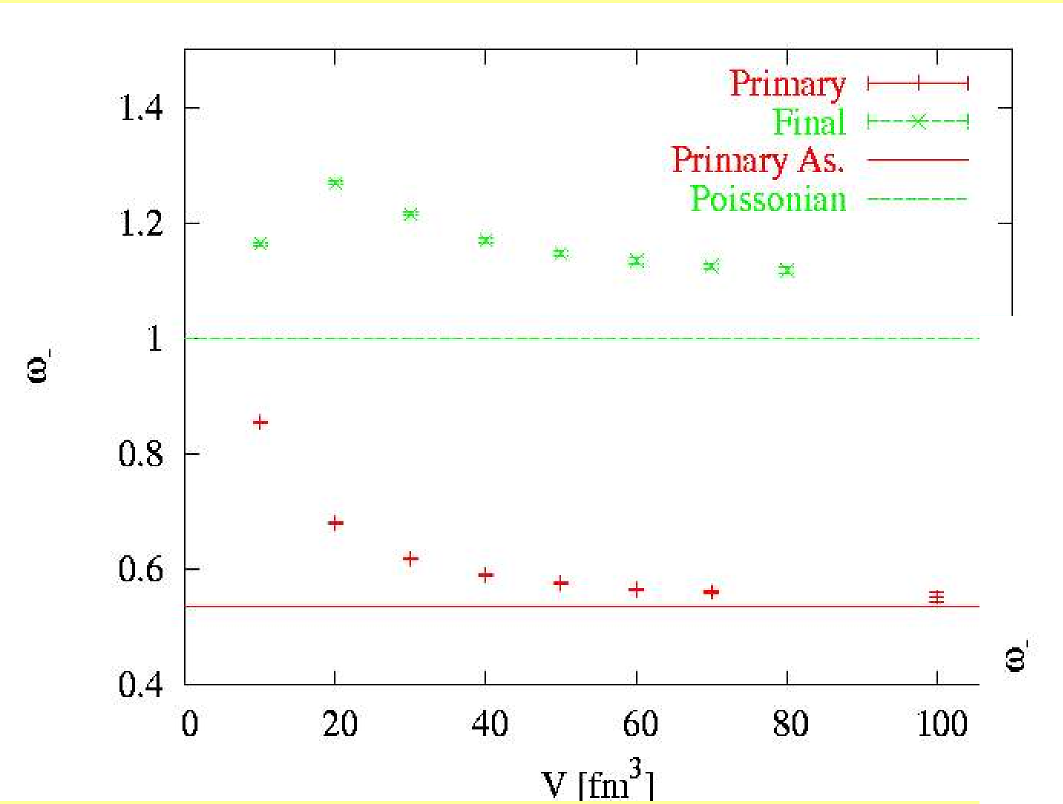
Total primaries



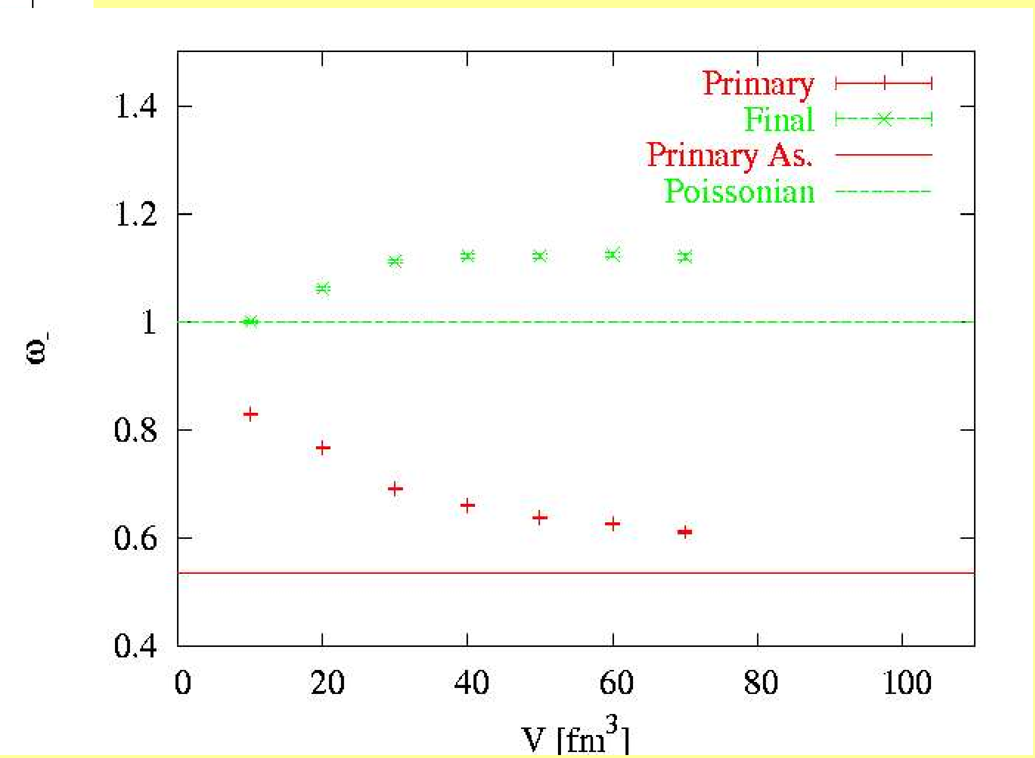
Charged after strong decay

Simulations

- Scaled variances of negatives in systems with T=160 MeV



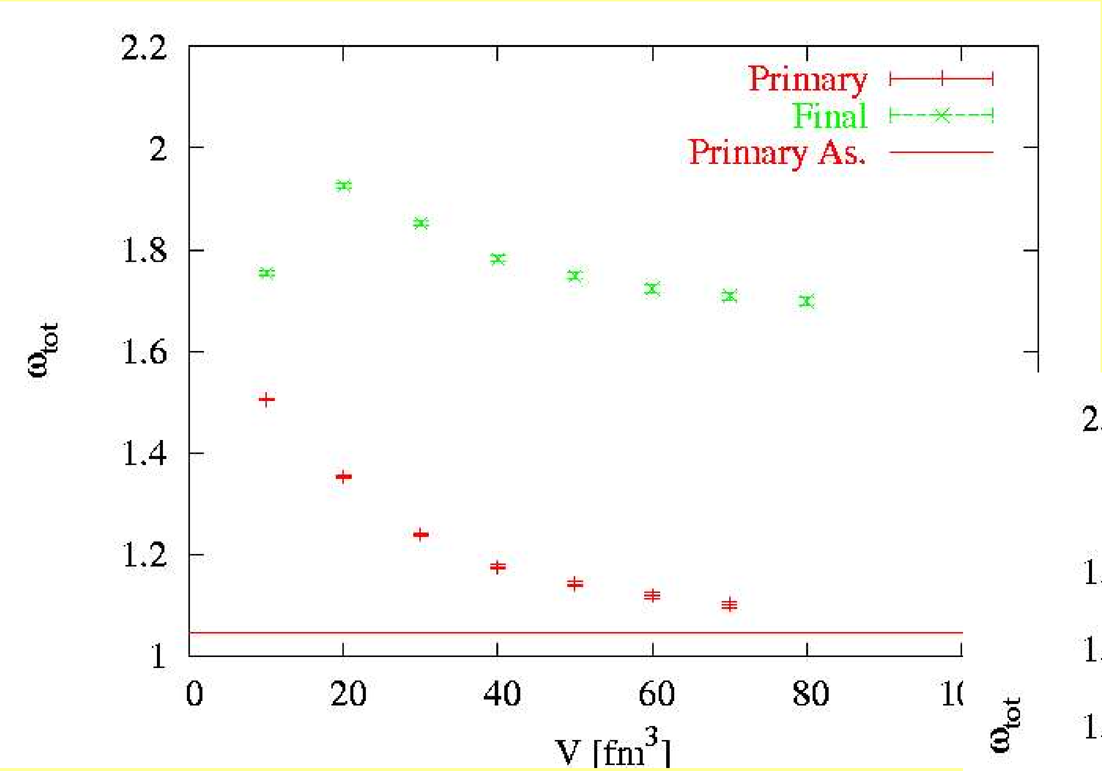
e^+e^- -like system



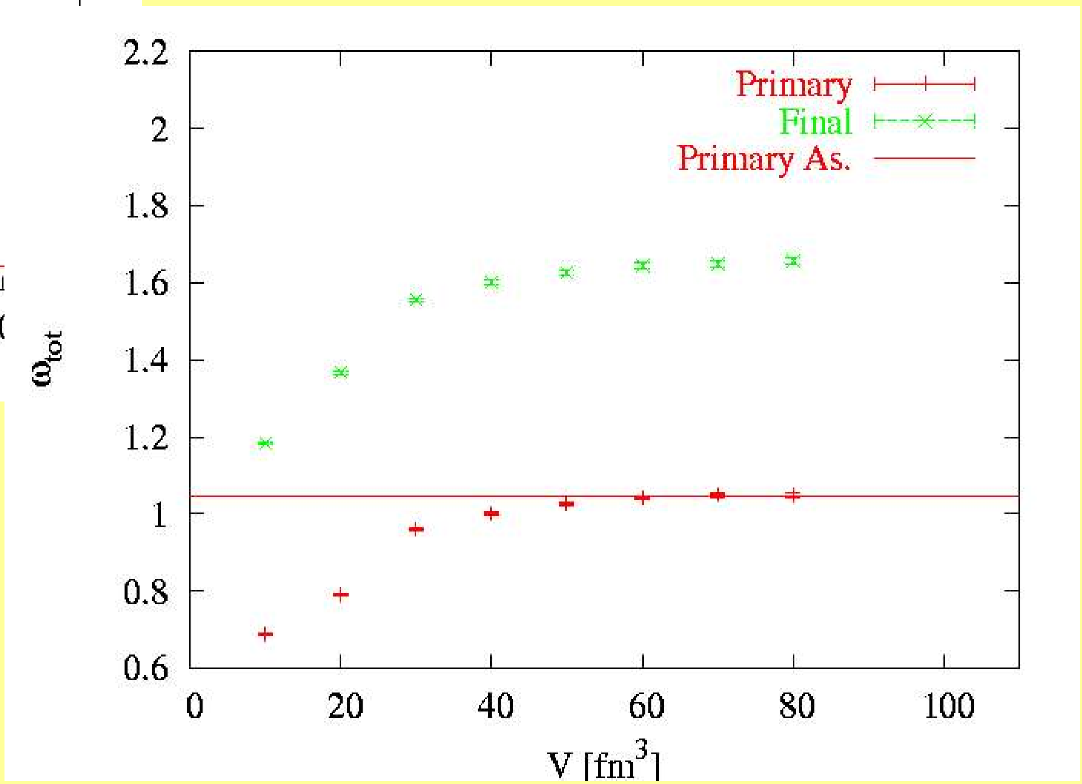
pp -like system

Simulations

- Scaled variances of total multiplicity in systems with T=160 MeV



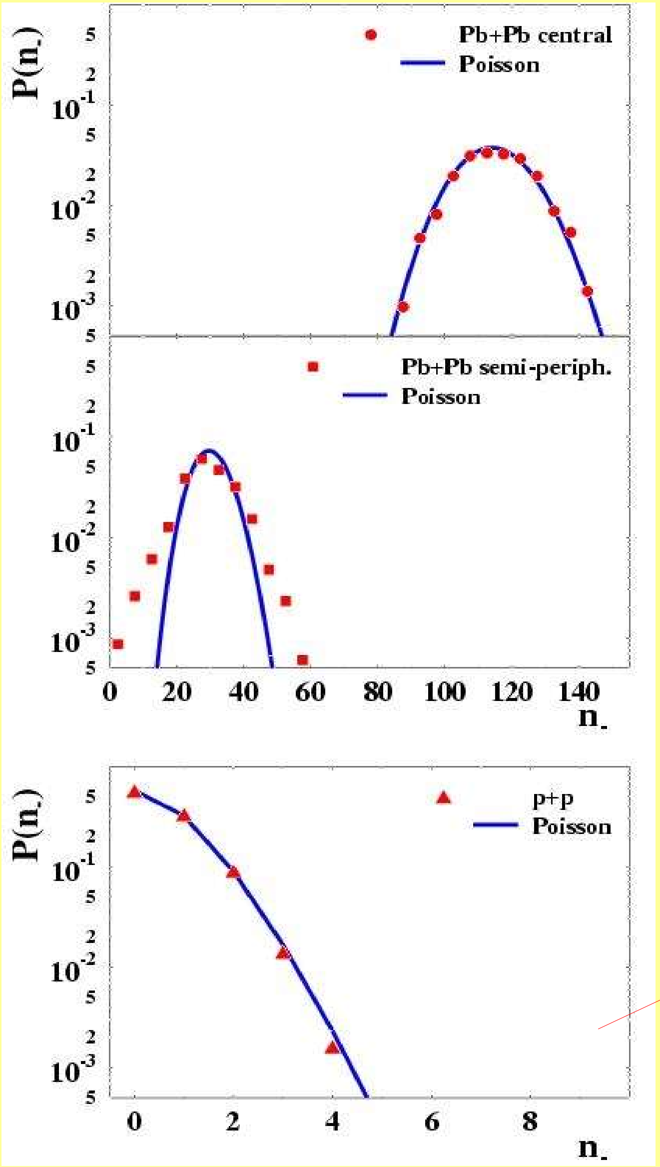
e^+e^- -like system

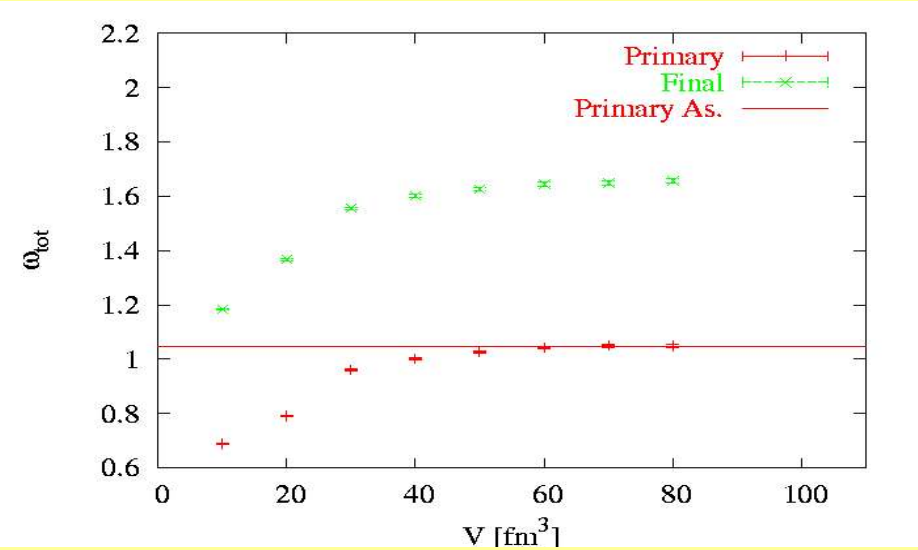


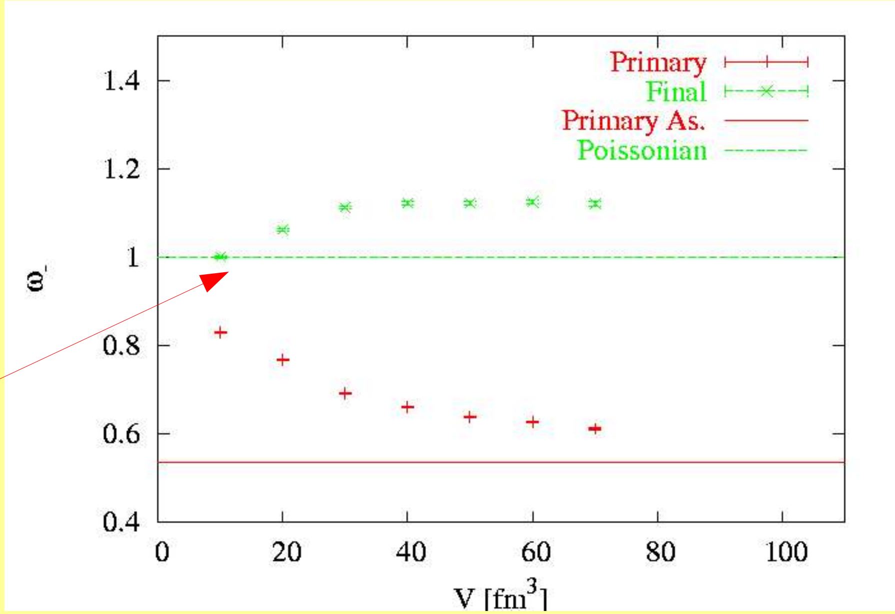
pp -like system

Simulations and Data

NA49 Prreliminary (cheers, Blume!)



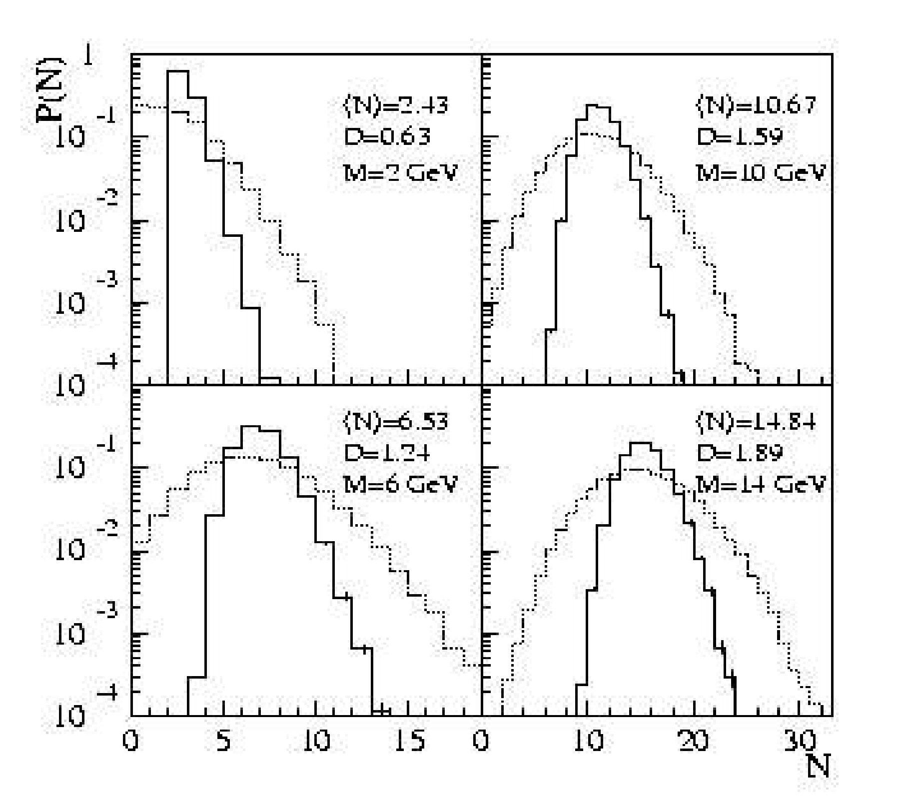




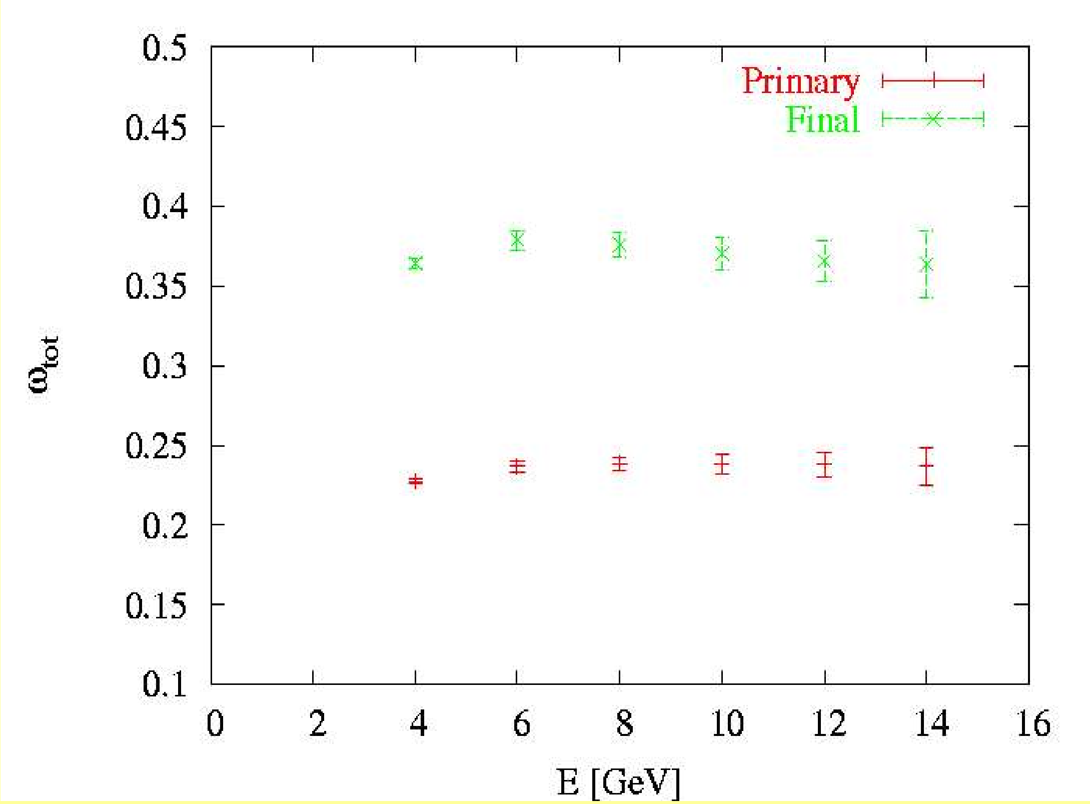
The Next Step: Microcanonical

- In addition to exact charges, force energy-momentum conservation

  Becattini & Ferroni, hep-ph/0407117



e^+e^- -like system, total primaries

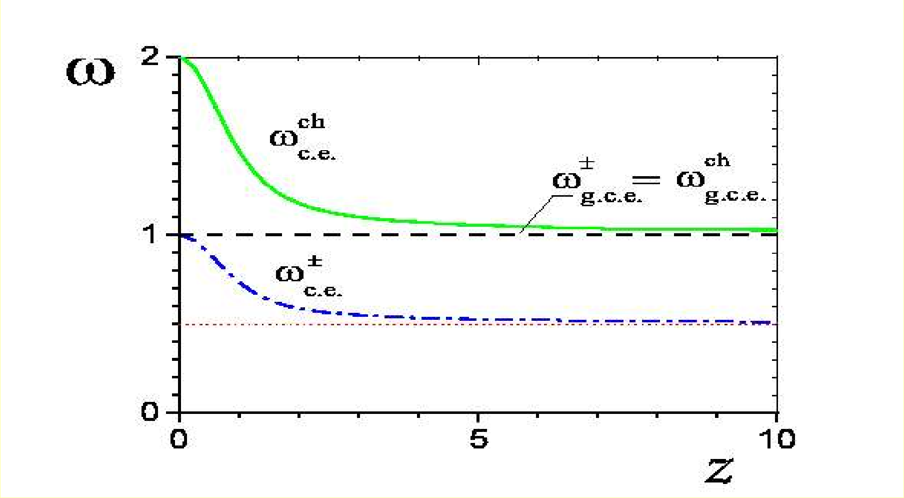


Much narrower than canonical!

Summary

• **Canonical relativistic gas:** *Even though averages approach Grand Canonical limit, dispersions have drastically different limits*

• **Pion gas:**



• **General asymptotic** *behaviour very similar to simple pion gas!*

• **Simulations:** *General V behaviour again similar to pion gas, spectra after strong decay much wider*

• **Micro Canonical simulations:** *Another ½ away from canonical scaled variance!*